ORLICZ FUNCTION SPACES AND COMPOSITION OPERATOR

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DECLARATION

I hereby declare that the project report entitled **"ORLICZ FUNCTION SPACES AND COMPOSITION OPERATOR"** submitted for the M.Sc. Degree is a review work carried out by me and the project has not formed the basis for the award of any Degree, Associateship, Fellowship or any other similar titles.

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CERTIFICATE

This is to certify that the work contained in this report entitled **"ORLICZ FUNCTION SPACES AND COMPOSITION OPERATOR"** submitted by **Chinmay Kumar Giri** (**Roll No: 411MA2075.**) to Department of Mathematics, National Institute of Technology Rourkela for the partial fulfilment of requirements for the degree of master of science in Mathematics towards the requirement of the course **MA592 Project** is a bonafide record of review work carried out by him under my supervision and guidance. The contents of this project, in full or in parts, have not been submitted to any other institute or university for the award of any degree or diploma.

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Contents

1 Introduction **2**

2 Orlicz Function Spaces **4**

2.1 Orlicz Spaces 4

3 Composition Operators On Orlicz Spaces **15**

3.1 Modular and norm continuity of composition operators 20

3.2 Compact Composition Operators in Orlicz space 23

English Symbols

$\mathbb{N}$	set of natural number .
$\mathbb{R}$	set of real number .
$\mathbb{C}$	set of complex number .
$\mathbb{K}$	field of scalars i.e. either $\mathbb{C}$ or $\mathbb{R}$.
X	vector space over the field $\mathbb{K}$.
Σ	sigma algebra
μ	measure defined over Σ.
Ω	$\sigma-$finite complete measure space.
Φ	Young's function.
$\|.\|_{\Phi}$	$\inf\{\lambda > 0 : \int_{\Omega}\Phi(\|\frac{x}{\lambda}\|)d\mu \leq 1\}$.
$L^{\Phi}(\Omega)$	Orlicz function space.
τ	nonsingular measurable transformation from Ω to itself.
C_{τ}	composition operator form$L^{\Phi}(\Omega)$ to itself generated by τ.
$\nu \ll \mu$	ν is absolutely continuous with respect to μ.
$\|.\|_{e}$	essential norm of a bounded linear operator.

Chapter 1

Introduction

Orlicz spaces have their origin in the Banach space researches of 1920. Indeed, after the development of Lebesgue theory of integration and inspired by the function t^p in the definitions of the spaces l^p and L^p, Orlicz spaces were first proposed by Z.W.Birnbaum and W.Orlicz in[1] and latter developed by Orlicz himself in[7], [8]. The study and applications of this theory was picked up again in Poland, USSR and Japan after the war years. Around the year 1950, H.Nakano [6] studied Orlicz spaces with the name "modulared spaces". However, the theory became popular for researches in the western countries after the publication of the book on "Linear Analysis" by A.C.Zaanen. This possibly resulted in the translation of the monograph of M.A.Krasnoselskii and Ya.B.Rutickii on Convex Function and Orlicz Spaces by Leo F. Boron from Russian to English, and after the appearance of English version of this book in 1961, the theory has been effectively used in many branches of Mathematics and Statistics e.g, differential and integral equations, harmonic analysis, probability etc.

Prior to the researches of W.Orlicz ,it was W.H.Young [12] who, motivated by the functions $u^p(u > 0)$ and $v^p(v > 0)$ with $\frac{1}{p} + \frac{1}{q} = 1$,$1 < p, q < \infty$, introduced a function $v = \Phi(u)$ for $u \geq 0$ such that Φ is continuous and strictly increasing with $\Phi(0) = 0$ and $\Phi(u) \to \infty$ as $u \to \infty$. if $u = \Psi(v)$ is the inverse of Φ, he defined

$$\Phi(a) = \int_0^a \Phi(u)du \text{ , } \Psi(b) = \int_0^b \Psi(v)dv$$

for $a, b \geq 0$. These functions are known as Young's function in the literature, and besides being convex, satisfy the Young's inequality

$$ab \leq \Phi(a) + \Psi(b)$$

for $a, b \geq 0$. Young introduced the classes Y_Φ and Y_Ψ consisting of measurable functions f for which $\int \Phi(|f(x)|)dx < \infty$ and $\int \Psi(|f(x)|)dx < \infty$, respectively. These spaces failed to form the vector space. However, if satisfies Δ_2-condition in the sense that there exists a constant $C > 0$ such that $\Phi(2u) \leq C\Phi(u)$ hold for all $u \geq 0$, Y_Φ becomes a vector space. In the process of norming the spaces Y_Φ, Y_Ψ, Orlicz considered the class L^Φ of all measurable functions f satisfying

$$\|f\|_\Phi = \sup\{\int |fg|dx : \int \Psi(|g|)dx \leq 1\} < \infty,$$

and proved that $(L^\Phi, \|\|_\Phi)$ is a normed linear space. In general, $Y_\Phi \subset L^\Phi$, however, if Φ satisfies Δ_2-condition defined as above, $Y_\Phi = L^\Phi$, cf.[9], [10].

In Mathematics, the composition operator C_Φ with symbol Φ is a linear operator defined with the help of composition of mapping $f \circ \Phi$ by the formula $C_\Phi(f) = f \circ \Phi$. Most of the recent interest in composition operators arises from the study of boundedness, compactness of these operators (see for example [10],[11]). In analysis this operator has of lost of connection with the Hardy space, space of analytic functions, L^p spaces, for $p \geq 1$.

The material is divided into three chapters. In chapter 2, the basic theory of Orlicz spaces is presented. Next, chapter 3 contains some results of composition operator on Orlicz spaces i.e. it's boundedness and compactness.

Chapter 2

Orlicz Function Spaces

This chapter includes some basic definitions and results which have been used in the next chapter. We present have the salient features from the theory of Orlicz function spaces, $L^{\Phi}(\Omega)$, generated by the Young's function Φ on an arbitrary $\sigma-$finite measurable spaces Ω. We will discussed these are the Banach spaces equipped with the equivalent orlicz and gauge norms.

2.1 Orlicz Spaces

Before going to the main results of this chapter, let us begin with the following definitions.

Definition 2.1.1. *A real function Φ defined on an interval (a, b), where $-\infty \leq a < b \leq \infty$ is called **convex** if the following inequality hold*

$$\Phi((1-\lambda)x + \lambda y) \leq (1-\lambda)\Phi(x) + \lambda\Phi(y)$$

whenever $a < x < b$, $a < y < b$ and $0 \leq \lambda \leq 1$.

Definition 2.1.2. *Let $\Phi : \mathbb{R} \to \mathbb{R}^+$ be a convex function such that*

i) $\Phi(-x) = \Phi(x)$

ii) $\Phi(x) = 0$ *iff x=0*

iii) $lim_{x\to\infty}\Phi(x) = \infty$.

Such a function Φ *is known as a* ***Young function***. $cf.$[9]

Example 2.1.1. i) $\Phi_p(s) := \frac{|s|^p}{p}$ with p≥ 1;

ii) Let $\Phi(x) = |x|^p$, $p \geq 1$. Then Φ is a continuous Young function such that $\Phi(x) = 0$ if and only if $x = 0$, and $\Phi(x) \to \infty$ as $x \to \infty$ while $\Phi(x) < \infty$ for all $x \in \mathbb{R}$.

Definition 2.1.3. *A function is called* ***N-function*** *if it admits the representation*

$$M(u) = \int_0^u p(t)dt$$

Where p(t) is right continuous for t ≥ 0, *positive for t* > 0 *and non decreasing which satisfies the condition* $p(0) = 0$ *and* $p(\infty) = \lim_{t\to\infty} p(t) = \infty$.

Example 2.1.2. The function $M(u) = \frac{|u|^\alpha}{\alpha}$ for $\alpha > 1$ is a N-function for $p(t) = t^{\alpha-1}$.

Definition 2.1.4. *We say that N-function M(u) satisfies the* Δ_2 *condition for the large values of u if there exists constant* $k > 0$, $u_0 \geq 0$ *such that*

$$M(2u) \leq kM(u),\ (u \geq u_0)$$

Definition 2.1.5. *Let* $\Omega = (\Omega, \Sigma, \mu)$ *be a* σ*-finite measurable space and let* $\tau : \Omega \to \Omega$ *be a* ***measurable transformation***, *that is* $\tau^{-1}(A) \in \Sigma$ *for any* $A \in \Sigma$. *If* $\mu(\tau^{-1}(A) = 0$ *for all* $A\epsilon\ \Sigma$ *with* $\mu(A)$=0, *then* τ *is said to be nonsingular.*

Definition 2.1.6. *An* ***atom*** *of the measure* μ *is an element* $A\in \Sigma$ *with* $\mu(A) > 0$ *such that for each* $F \in \Sigma$,*if* $F \subset A$ *then either* $\mu(F) = 0$ *or* $\mu(F) = \mu(A)$.

Definition 2.1.7. *A set $A \in \Sigma$ is an atom for μ if $\mu(A) > 0$ and for each $B \subset A, B \in \Sigma$ either $\mu(B) = 0$ or $\mu(A - B) = 0$. A set $D \in \Sigma$ is diffuse for μ if it does not contain any$\mu-$atom.i.e. for $0 \leq \lambda \leq \mu(D)$ we can find a set $D_1 \subset D, D_1 \in \Sigma$ such that $\mu(D_1) = \lambda$.*

Definition 2.1.8. *Let $\tilde{L}^{\Phi}(\Omega)$ be the set of all $f : \Omega \to R$, measurable for Σ, such that $\int_{\Omega} \Phi(|f|)d\mu < \infty$.*

Theorem 2.1.1. 1. *The space $\tilde{L}^{\Phi}(\Omega)$ introduced above is absolutely convex,i.e. if $f, g \in \tilde{L}^{\Phi}(\Omega)$ and α, β are scalars such that $|\alpha| + |\beta| \leq 1$, then $\alpha f + \beta g \in \tilde{L}^{\Phi}(\Omega)$. Also $h \in \tilde{L}^{\Phi}(\Omega), |f| \leq |h|$, f measurable $\Rightarrow f \in \tilde{L}^{\Phi}(\Omega)$.*

2. *The space $\tilde{L}^{\Phi}(\Omega)$ is linear space if $\Phi \in \Delta_2$ globally when $\mu(\Omega) = \infty$, and locally if $\mu(\Omega) < \infty$ and Δ_2-condition is necessary if μ is diffuse on a set of positive measure.*

Proof. 1. let $f, g \in \tilde{L}^{\Phi}(\Omega)$ and α, β are scalars such that $|\alpha|+|\beta| \leq 1$. let $\gamma = |\alpha|+|\beta| \leq 1$. Then by using the monotonicity and convexcity of Φ we get $\Phi(|\alpha f + \beta g|) \leq \Phi(|\alpha||f| + |\beta||g|) \leq \gamma\Phi(\frac{|\alpha|}{\gamma}|f| + \frac{|\beta|}{\gamma}|f|) = \gamma.\frac{|\alpha|}{\gamma}\Phi(|f|) + \gamma.\frac{|\beta}{\gamma}\Phi(|g|) = |\alpha|\Phi(|f|) + |\beta|\Phi(|g|)$, but by hypothesis right hand side is integrable. Hence $\alpha f + \beta g \in \tilde{L}^{\Phi}(\Omega)$. Since Φ is monotonically increasing and $|f| \leq |h|$ hence $\Phi(|f|) \leq \Phi(|h|)$.

2. To prove $\tilde{L}^{\Phi}(\Omega)$ is a vector space it is sufficient to prove, for each $f \in \tilde{L}^{\Phi}(\Omega), 2f \in \tilde{L}^{\Phi}(\Omega)$ then $nf \in \tilde{L}^{\Phi}(\Omega)$ for any $n \in \mathbb{N}$ and then for each $\alpha > 0, \alpha f \in \tilde{L}^{\Phi}(\Omega)$. Let a, b be any scalars, let $\gamma = |a| + |b| > 0$ then we have $af + bg = \gamma(\frac{a}{\gamma}f + \frac{b}{\gamma}g) \in \tilde{L}^{\Phi}(\Omega)$ for any $f, g \in \tilde{L}^{\Phi}(\Omega)$. Now only remain to prove for each $f \in \tilde{L}^{\Phi}(\Omega, 2f \in \tilde{L}^{\Phi}(\Omega)$.
Since $\Phi \in \Delta_2$ globally, then we have $\mu(\Omega) = \infty, \Phi(2|f|) \leq K\Phi(|f|), K > 0 \Rightarrow \int_{\Omega} \Phi(2|f|) \leq K \int_{\Omega} \Phi(|f|) < \infty$ hence we have $2f \in \tilde{L}^{\Phi}(\Omega)$.
Let $\Phi \in \Delta_2$ locally, then we have $\mu(\Omega) < \infty$, then $\Phi(2x) \leq K\Phi(x)$ for each $0 \leq x_0 \leq x$. Now let $f_1 = f$ if $|f| \leq x_0$ and 0 otherwise. Let $f_2 = f - f_1$ so that $f = f_1 + f_2$ and

$$\Phi(2|f|) = \Phi(2|f_1|) + \Phi(2|f_2|) \leq \Phi(2|f_1|) + K\Phi(|f_2|).$$

Hence

$$\int_\Omega (2|f|)d\mu \le \Phi(2x_0)\mu(\Omega) + K\int_\Omega \Phi(|f|)d\mu < \infty$$

,

Thus $2f \in \tilde{L}^\Phi(\Omega)$. Hence $\tilde{L}^\Phi(\Omega)$ is a vector space when $\Phi \in \Delta_2$.

Conversely, let $E \in \Sigma$ be a set of positive measure and let μ diffuse on E and $\Phi \in \Delta_2$ is not regular.

To prove necessity of Δ_2 condition we have to construct a $f \in \tilde{L}^\Phi(\Omega)$ such that $2f \notin \tilde{L}^\Phi(\Omega)$. Let assume that $0 < \alpha < \mu(E) \le \infty$. Then by given statement on μ,there is an $F \subset E, F \in \Sigma$ with $\mu(F) = \alpha$. Since $\Phi \notin \Delta_2$, $\exists$ a sequence $x_n \ge n$ such that $\Phi(2x_n) \ge n\Phi(x_n), n \ge 1$. Let $n_0 \in \mathbb{N}$ such that

$$\sum_{n\ge n_0} \frac{1}{n^2} < \alpha \text{ and } \Phi(x_n) \ge 1 \text{ for all } n \ge n_0.$$

Since μ is diffuse on F, there is a measurable $F_0 \subset F$ such that $\mu(F_0) = \sum \frac{1}{n^2} < \alpha$. We can find a set $D_1 \in \Sigma, D_1 \subset F_0$ such that $\mu(D_1) = 1/n_0^2$. Similarly again we can find set $D_2 \in \Sigma, D_2 \subset F_0 - D_1$ such that $\mu(D_2) = 1/(n_0+1)^2$. On repeating this process, we can find disjoint sets $D_n \in \Sigma$ such that $\mu(D_n) = 1/(n_0+n-1)^2, n \ge 1$.

Let $F_k \subset D_k, F_k \in \Sigma$ such that $\mu(F_k) = \frac{\mu(D_K)}{\Phi(x_n)}$.

Let $f = \sum_{n=1}^{\infty} x_n \chi_{F_n}$ then clearly f is measurable. Now

$$\begin{aligned}\int_\Omega \Phi(|f|)d\mu &= \sum_{n=1}^{\infty} \Phi(x_n)\mu(F_n) \\ &= \sum_{n\ge n_0} 1/n^2 < \infty\end{aligned}$$

so $f \in \tilde{L}^\Phi(\Omega)$.

Now

$$\begin{aligned}\int_\Omega \Phi(2f)d\mu &= \sum_{n=1}^{\infty}\Phi(2x_n)\mu(F_n)\\ &\geq \sum_{n\geq n_0} n\Phi(x_n)\mu(F_n)\\ &= \sum_{n\geq n_0}\frac{1}{n}=\infty\end{aligned}$$

.

so $2f \notin \tilde{L}^{\Phi}(\Omega)$ So Δ_2 condition is necessary to prove $\tilde{L}^{\Phi}(\Omega)$ is a vector space.

□

Example 2.1.3. Let $\Omega = \{1, 2, ...\}, \Sigma =$ the power set of Ω, and μ be the counting measure, i.e., $\mu(\{i\}) = 1$, $i \geq 1$. Let $\Phi(x) = e^{x^2} - 1$. Then Φ is a N-function and $\Phi \in \Delta_2$. We assume that $\tilde{L}^{\Phi}(\Omega)$ is a linear space. In fact, if f$\in \tilde{L}^{\Phi}(\Omega)$ then

$$\int_\Omega \Phi(f)d\mu = \sum_{n=1}^{\infty}(e^{|f(n)|^2} - 1) < \infty$$

So terms of right hand side are bounded. Let $K > 0$ be the bound so that the $e^{(|f(n)|^2)} \leq K + 1, n \geq 1$. Then

$$\begin{aligned}\int_\Omega \Phi(2f)d\mu &= \sum_{n=1}^{\infty}(e^{(4|f(n)|^2)} - 1)\\ &\leq \sum_{n=1}^{\infty}(e^{|f(n)|^2} - 1)(K+2)((K+1)^2+1)\\ &= (K+2)((K+1)^2+1)\int_\Omega \Phi(f)d\mu < \infty.\end{aligned}$$

Hence $2f \in \tilde{L}^{\Phi}(\Omega)$, and the space is linear.

Definition 2.1.9. *Let $\Phi : [0,\infty) \to [0,\infty)$ be a continuous, non-decreasing and convex function with $\Phi(0) = 0$, $\Phi(x) > 0$ for $x > 0$ and $\Phi(x) \to \infty$ as $x \to \infty$. Such function is known as an* ***Orlicz function****.*

Definition 2.1.10. *Let $L^{\Phi}(\Omega)$ be the set of all measurable functions such that $\int_{\Omega} \Phi(\alpha|f|)d\mu < \infty$ for some α >0. The space $L^{\Phi}(\Omega)$ is called **Orlicz Space**.*

Thus $L^{\Phi}(\Omega) = \{f : \Omega \to [0, \infty]$, measurable: $\int_{\Omega} \Phi(\alpha|f|)d\mu < \infty$ for some $\alpha > 0\}$.

Theorem 2.1.2. *The set $L^{\Phi}(\Omega)$ is a vector space.*

Proof. Let $f_1, f_2 \in L^{\Phi}(\Omega)$.

Then there exist $\alpha_1 > 0$ and $\alpha_2 > 0$ such that $\int_{\Omega} \Phi(\alpha_1|f|)d\mu < \infty$ and $\int_{\Omega} \Phi(\alpha_2|f|)d\mu < \infty$.

Let $\alpha = min\{\alpha_1, \alpha_2\}$, then $\alpha > 0$.

Now by using convexcity of Φ we get

$$\int_{\Omega} \Phi(\tfrac{\alpha}{2}(f_1 + f_2))d\mu \leq \tfrac{1}{2} \int_{\Omega} \Phi(\alpha_1 f_1)d\mu + \tfrac{1}{2} \int_{\Omega} \Phi(\alpha_2 f_2)d\mu.$$

Hence $\int_{\Omega} \Phi(\frac{\alpha}{2}(f_1 + f_2))d\mu < \infty$, Where $\frac{\alpha}{2} > 0$.

Hence $f_1 + f_2 \in L^{\Phi}(\Omega)$. Thus $L^{\Phi}(\Omega)$ is closed under addition.

Now we have to prove $L^{\Phi}(\Omega)$ is closed under scalar multiplication.

Let $f \in L^{\Phi}(\Omega) \Rightarrow f + f = 2f \in L^{\Phi}(\Omega)$, hence $nf \in L^{\Phi}(\Omega)$ for all integers $n > 1$. Now for any $\beta \in \mathbb{R}$ there exists $n_0 \in \mathbb{N}$ such that $|\beta| \leq n_0 \Rightarrow |\beta f| \leq |n_0 f|$. So by theorem 2.1.1 $\beta f \in L^{\Phi}(\Omega)$. Hence $L^{\Phi}(\Omega)$ is a vector space. □

Definition 2.1.11. *If $f, g \in \tilde{L}^{\Phi}(\Omega)$ and α, β are scalars such that $|\alpha| + |\beta| \leq 1$ then $\alpha f + \beta g \in \tilde{L}^{\Phi}(\Omega)$. Also $h \in \tilde{L}^{\Phi}(\Omega)$, $|f| \leq |h|$, f is measurable then $f \in \tilde{L}^{\Phi}(\Omega)$. Any space of function with the above property is called **circled and solid space**.*

Lemma 2.1.1. *Let*

$$\mathbf{B_{\Phi}} = \{g \in \tilde{L}^{\Phi}(\Omega) : \textstyle\int_{\Omega} \Phi(g)d\mu \leq 1\},$$

and $\mathbf{B_{\Phi}}$ is a circled and solid subset of $\tilde{L}^{\Phi}(\Omega)$ and $f \in L^{\Phi}(\Omega)$ if and only if $\alpha f \in \mathbf{B_{\Phi}}$ for some $\alpha > 0$.

Proof. Let $f, g \in \mathbf{B}_\Phi$ and α, β are scalars such that $|\alpha| + |\beta| \leq 1$. Then

$$\begin{aligned}\int_\Omega \Phi(|\alpha f + \beta g|)d\mu &\leq |\alpha| \int_\Omega \Phi(|f|)d\mu + |\beta| \int_\Omega \Phi(|g|)d\mu \\ &\leq |\alpha| + |\beta| \\ &\leq 1\end{aligned}$$

Let $f \in L^\Phi(\Omega)$ so that $\alpha f \in \tilde{L}^\Phi(\Omega)$ for some $\alpha > 0$. Let $a_n \searrow 0$ be arbitrary and let $\alpha_n = min\{\alpha, a_n\}$. Then $\alpha_n \searrow 0$ and $\Phi(\alpha_n f) \leq \Phi(\alpha f)$ and $\Phi(\alpha_n f) \to 0$ when Φ is a continuous Young function. Hence by dominated convergence then $\int_\Omega(\alpha_n f)d\mu \to 0$ so that there is some $n_0 \in \mathbb{N}$ such that $\int_\Omega \Phi(\alpha_{n_0} f)d\mu \leq 1$. Thus $\alpha_{n_0} f \in \mathbf{B_\Phi}$. □

Definition 2.1.12. *For $f \in L^\Phi(\Omega)$ define $\|f\|_\Phi = inf\{\lambda > 0 : I_\Phi(\frac{f}{\lambda}) \leq 1\}$, where $I_\Phi(\frac{f}{\lambda}) = \int_\Omega \Phi(|\frac{f}{\lambda}|)d\mu$ is called the modular of Φ.*

Theorem 2.1.3. *$(L^\Phi(\Omega), \|.\|_\Phi)$, is a normed linear space.*

Proof. Clearly $L^\Phi(\Omega)$ is a linear space.

Next we have to verify $\|.\|_\Phi$ is a norm on $L^\Phi(\Omega)$, i.e. to verify the following three conditions.

(i) $\|f\|_\Phi = 0$ iff $f = 0$ a.e.

(ii) $\|\alpha f\|_\Phi = |\alpha|\|f\|_\Phi$ for all $\alpha \in$ K

(iii) $\|f + g\|_\Phi \leq \|f\|_\Phi + \|g\|_\Phi$

Clearly if $f = 0$ a.e. then $\|f\|_\Phi = 0$.

Conversly, Let $\|f\|_\Phi = 0$, to show that $f = 0$ a.e. If possible let $|f| > 0$ on a set of positive measure. Then there exists a number $\delta > 0$ such that $A = \{x : |f(x)| \geq \delta\}$ satisfies

$\mu(A) > 0 \Rightarrow \frac{f}{k} \in \mathbf{B}_\Phi$ for all $k > 0 \Rightarrow nf \in \mathbf{B}_\Phi$ for all $n \geq 1$. Hence, for $n \geq 1$

$$\begin{aligned}
\Phi(n\delta)\mu(A) &= \int_A \Phi(n\delta)d\mu \\
&\leq \int_A \Phi(nf)d\mu \\
&\leq \int_\Omega \Phi(nf)d\mu \\
&\leq 1
\end{aligned}$$

Since $\mu(A) > 0$, we have $\Phi(n\delta) \to \infty$ as $n \to \infty$ which is a contradiction. Hence $f = 0$ a.e. for (ii) consider the non-trivial case $\alpha \neq 0$.

$$\begin{aligned}
\|\alpha f\|_\Phi &= \int_\Omega \Phi(|\frac{\alpha x}{\lambda}|)d\mu \leq 1 \\
&= |\alpha| inf\{\frac{\lambda}{|\alpha|} > 0 : \int_\Omega \Phi(|\frac{f}{\frac{\lambda}{|\alpha|}}|)d\mu \leq 1\} \\
&= |\alpha| inf\{\beta > 0 : \int_\Omega \Phi(\beta|f|)d\mu \leq 1\} \\
&= |\alpha|. \parallel f \parallel_\Phi
\end{aligned}$$

finally for triangle inquality (iii),by definition of infimum there exists $\alpha_1 > 0$ and $\alpha_2 > 0$ such that $\|f_1\|_\Phi < \alpha_1 + \frac{\epsilon}{2}$ and $\|f_2\|_\Phi < \alpha_2 + \frac{\epsilon}{2}$.

Let $\beta = \alpha_1 + \alpha_2$

Since $f_1 + f_2 \in L^\Phi(\Omega), \|f_1 + \|f_2 < \infty$.

consider $\int_\Omega \Phi((f_1 + f_2)/\beta)d\mu = \int_\Omega \Phi(\frac{f_1}{\alpha_1}.\frac{\alpha_1}{\beta} + \frac{f_2}{\alpha_2}.\frac{\alpha_2}{\beta})d\mu$

$\leq \frac{\alpha_1}{\beta} \int_\Omega \Phi(\frac{f_1}{\alpha_1})d\mu + \frac{\alpha_2}{\beta} \int_\Omega \Phi(\frac{f_2}{\alpha_2})$(by using convexity of Φ)

$\leq \frac{\alpha_1}{\beta} + \frac{\alpha_2}{\beta} = 1$

$\Rightarrow \frac{1}{\beta}(f_1 + f_2) \in L^\Phi(\Omega)$

Hence $\|\frac{1}{\beta}(f_1+f_2)\|_\Phi = \frac{1}{\beta}\|(f_1+f_2)\|_\Phi \leq 1$

$\Rightarrow \|(f_1+f_2)\|_\Phi \leq \beta$

But $\beta = \alpha_1+\alpha_2, \|f_1\| < \alpha_1 + \frac{\epsilon}{2} and \|f_2\| < \alpha_2 + \frac{\epsilon}{2}$

$\Rightarrow \|f_1\|_\Phi + \|f_2\|_\Phi < \alpha_1+\alpha_2+\epsilon$

$\Rightarrow \|f_1\|_\Phi + \|f_2\|_\Phi < \beta+\epsilon$

$\Rightarrow \|f_1\|_\Phi + \|f_2\|_\Phi < |(f_1+f_2)\|_\Phi + \epsilon$

Since $\epsilon > 0$ be arbitrary

$\Rightarrow \|f_1\|_\Phi + \|f_2\|_\Phi \leq \|(f_1+f_2)\|_\Phi$

Hence (iii) follows.

Thus $\|x\|_\Phi = inf\{\lambda > 0 : \int_\Omega \Phi(|\frac{x}{\lambda}|)d\mu \leq 1\}$ is the norm defined on $L^\Phi(\Omega)$. Hence $L^\Phi(\Omega)$ is a normed linear space. □

Remark 2.1.1. *The above norm $\|.\|_\Phi$ defined on the space $L^\Phi(\Omega)$ is known as* ***Luxemburg-Nakano Norm****.*

Lemma 2.1.2. *$\|f\|_\Phi \leq 1$ if and only if $\int_\Omega \Phi(f)d\mu \leq 1$.*

Proof. Let $\alpha = \|f\|_\Phi, f \in L^\Phi(\Omega)$. If $\alpha = 0$ then it is trivial so let $\alpha > 0$. Then by definition, $\frac{1}{\alpha} \in \mathbf{B}_\Phi$. If $\alpha \leq 1$,then

$$\int_\Omega \Phi(f)d\mu \leq \int_\Omega \Phi(\frac{f}{\alpha})d\mu \leq 1$$

so that $\|f\|_\Phi \leq 1$ implies that left hand side is bounded by 1 on other hand, $f \in \mathbf{B}_\Phi$ then by definition $\|f\|_\Phi \leq 1$ holds. □

Remark 2.1.2. *If $\alpha > 1$ then $\int_\Omega \Phi(\frac{f}{\alpha})d\mu \leq 1$ but $\int_\Omega \Phi(f)d\mu = \infty$ is possible. Thus only $0 \leq \alpha \leq 1$ is possible here.*

Theorem 2.1.4. *The normed linear space $(L^\Phi(\Omega), \|.\|_\Phi)$ is a Banach Space.*

Proof. Let $\{f_n, n > 1\}$ be a cauchy sequence in $L^\Phi(\Omega)$ such that $\|f_n - f_m\|_\Phi \to 0$ as $m, n \to \infty$ we have to construct a $f \in L^\Phi(\Omega)$ which satisfy $\|f_n - f\|_\Phi \to 0$ as $n \to \infty$. Since we considered Φ is a Young function there are two cases.

Let $x_0 = sup\{x \in \mathbb{R}^+ : \Phi(x) = 0\}$. Then by definition of Φ the above set define in the braces is compact so $0 \leq x_0 < \infty$. Then by hypothesis there exist numbers $k_{m,n} \geq 0$ ($k_{m,n}^{-1} \leq \|f_n - f_m\|_\Phi$) such that

$$\int_\Omega \Phi(k_{m,n}|F_n - f_m|)d\mu \leq 1 \qquad (1)$$

Let define $A_{mn} = \{\omega : k_{m,n}|F_n - f_m|(\omega) > x_0\} \in \Sigma$ is at most $\sigma-$finite for μ. Let $B_k = B_k^{mn} = \{\omega : k_{m,n}|F_n - f_m|(\omega) > x_0 + k^{-1}\}$, then clearly $A_{mn} = \cup_{k=1}^\infty B_k$ and for each k $\mu(B_k) < \infty$. Since by condition(1)

$$\mu(B_k) \leq \frac{1}{\Phi(x_0 + k^{-1})} \int_{B_k} \Phi(k_{m,n}|F_n - f_m|)d\mu \leq 1 \qquad (2)$$

Hence each A_{mn} is $\sigma-$finite and let $A = \cup_{m,n\geq 1} A_{mn}$. Thus on $A^c, k_{m,n}|F_n - f_m|(\omega) \leq x_0$, so that $\omega \in A^c, |f_n(\omega) - f_m(\omega)| \to 0$ uniformly. Thus there is a measurable function g_0 on A^c such that $f_n(\omega) \to g_0(\omega)$, and $|g_0(\omega)| \leq x_0, \omega \in A^c$.

Let us take Ω for A then $\{f_n\}$ is a cauchy sequence on $L^\Phi(\Omega)$ and hence for each $B \in \Sigma, \mu(B) < \infty$ by condition (2), we have

$$\begin{aligned}
\mu(B \cap \{|f_n - f_m| \geq \epsilon\}) &= \mu(B \cap \{\Phi(k_{mn}|f_n - f_m|) \geq \Phi(k_{mn}\epsilon)\}) \\
&\leq \frac{1}{\Phi(k_{mn}\epsilon)} \int_B \Phi(k_{m,n}|F_n - f_m|)d\mu \\
&\leq [\Phi(k_{mn}\epsilon)]^{-1}
\end{aligned}$$

Since $\epsilon > 0$ be fixed and and $k_{mn} \to \infty$, from we get that $\{f_n\}$ is a cauchy sequence in $\mu-$measure on each B by using the $\sigma-$finiteness property we have $\{f_n\}$ is a cauchy sequence in measure. If $f_n \to \tilde{f}$ in measure. Then there is subsequence $\{f_{n_i}\}$ such that then $f_{n_i} \to f$ a.e. But $\{f_{n_i}\}$ is a cauchy sequence in $\|f\|_\Phi$, we get $\|f_n\|_\Phi \to \rho$ and hence $\|f_{n_i}\|_\Phi \to \rho$. By using Fatou's lemma we get

$$\int_\Phi (\frac{|f|}{\rho}) \leq lim_{t\to\infty} inf \int_\Omega \Phi(\frac{f_{n_i}}{\|f_{n_i}\|_\Phi}) \leq 1$$

.

Hence $f \in L^\Phi(\Omega)$. Let m be fixed and $k \geq 0$ be given, then $\Phi(|f_{n_i} - f_{n_j}|k) \to \Phi(|f - f_{n_j}|k)$ as $i \to \infty$, a.e. let $n_i, n_j \geq n_0$ and $k_{n_i,n_j} \geq k$ then

$$\int_\Omega \Phi(k|F_{n_i} - f_{n_j}|)d\mu \leq \int_\Omega \Phi(k_{i,n_j}|F_{n_i} - f_{n_j}|)d\mu \leq 1$$

let $n_i \to \infty$ then by using Fatou's lemma we get $\|f - f_{n_j}\|_\Phi \leq \frac{1}{k}$. Since $k > 0$ is arbitrary $\|f - f_{n_j}\|_\Phi \to 0$ If f_{n_j} is any other subsequence with limit f', then $\{f_{n'_j}, f_{n_i}, i \geq 1, j \geq 1\} \subset \{f_n\}$ so that $f = f'$ a.e. because $f_n \to f$ in measure. So for every convergent subsequence and hence for the whole sequence,$\|f_n - f\|_\Phi \to 0$. This shows that every cauchy sequence of $(L^\Phi(\Omega), \|.\|_\Phi)$ converges to an element in the space. $\square$

Chapter 3

Composition Operators On Orlicz Spaces

This chapter is devoted to the study of Composition operators C_τ between Orlicz spaces $L^\Phi(\Omega)$ generated by measurable and non-singular transformations τ from Ω into itself. We also investigate the Boundedness and compactness of the composition operators on the Orlicz spaces by using different types of Δ_2 conditions of the orlicz function Φ.

Definition 3.0.13. *A measure μ on Ω is **complete** if whenever $E \in \Omega$, $F \subseteq E$ and $\mu(E) = 0$, then $F \in \Omega$.*

Definition 3.0.14. *A measure μ on Ω is $\sigma-$**finite** if for every set $E \in \Omega$,we have $E = \cup E_n$ for some sequence $\{E_n\}$ such that $E_n \in \Omega$ and $\mu(E_n) < \infty$ for each n.*

Example 3.0.4. *The Lebesgue measure m defined on $\mathbb{R}$,the class of mesurable sets of **R**, is $\sigma-$finite and complete.*

Definition 3.0.15. *If μ and ν are measures on the measure space (Ω, Σ) and $\nu(E) = 0$ whenever $\mu(E) = 0$, then we say that ν is **absolutely continuous** with respect to μ and we write $\nu \ll \mu$.*

Theorem 3.0.5. *If (Ω, Σ, μ) is a $\sigma-$finite measure space and ν is a $\sigma-$finite measure on Ω such that $\nu \ll \mu$, then there exists a finite-valued non-negative measurable function f on Ω*

such that for each E$\in \Sigma$, $\nu(E) = \int_E f d\mu$. *Also f is unique in the sense that if* $\nu(E) = \int_E g d\mu$ *for each* $E \in \Omega$, *then f=g a.e.*(μ).

Definition 3.0.16. *Let* μ *and* ν *be* $\sigma-$*finite measure on* (Ω, Σ) *and suppose that* $\nu \ll \mu$. *Then the* ***Radon-Nikodym derivative*** $\frac{d\nu}{d\mu}$, *of* ν *with respect to* μ, *is any measurable function f such that* $\nu(E) = \int_E f d\mu$ *for each* $E \in \Omega$.

Definition 3.0.17. *Let X and Y be two non-empty sets and let F(X) and F(Y) be two topological vector spaces of complex valued functions on X and Y respectively. Suppose T:* $X \to Y$ *is a mapping such that* $f \circ T \in F(Y)$ *whenever* $F \in F(X)$. *Then we define a composition transformation* $C_T : F(X) \to F(Y)$ *by* $C_T = f \circ T$ *for every* $f \in F(X)$.*If* C_T *is continuous, we call it a composition operators induced by T.*

Definition 3.0.18. *Let* $\mathbf{B}$ *be a Banach space and* $\mathbf{K}$ *be the set of all compact operators on* $\mathbf{B}$. *For* $T \in \mathbf{L}(\mathbf{B})$, *the Banach algebra of all bounded linear operators on* $\mathbf{B}$ *into itself, the essential norm of T means the distance from T to* $\mathbf{K}$ *in the operator norm,namely*

$$\|T\|_e = inf\{\|T - S\| : S \in \mathbf{K}\}.$$

Clearly, T is compact iff $\|T\|_e = 0$.

We need the following result for proving continuity of the composition operator.

Theorem 3.0.6. ***(Closed Graph Theorem)*** *Let X and Y be Banach spaces and* $F : X \to Y$ *be a closed linear map, then F is continuous.*

Theorem 3.0.7. *The composition map* $C_\tau : L^\Phi(\Omega) \to L^\Phi(\Omega)$ *is continuous.*

Proof. Let $\{f_n\}$ and $\{C_\tau f_n\}$ be sequence in $\mathrm{L}^\Phi(\Omega)$ such that $f_n \to f$ and $C_\tau f_n \to g$ for some $f, g \in L^\Phi(\Omega)$. Then we can find a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that

$$\Phi(|f_{n_k} - f|)(x) \to 0 \text{ for } \mu\text{−almost all } x \in \Omega.$$

from non-singularity of τ,

$$\Phi(|f_{n_k} - f|\tau)(x) \to 0 \text{ for } \mu- \text{ almost all } x \in \Omega.$$

From the above two relation, we conclude that $\mathrm{C}_\tau f = g$. This proved that the graph is closed and by using closed graph theorem C_τ is continuous. □

Theorem 3.0.8. *Let Ω_2 consists of infinitely many atoms, Φ be an Orlicz function and τ be a non-singular measurable transformation from Ω into itself. Put*

$$\alpha = inf\{\epsilon > 0 : N(h,\epsilon) \; consists \; of finitely \; many \; atoms\}$$

where $N(h,\epsilon) = \{x \in \Omega : h(x) > \epsilon\}$.If $C_\tau : L^\Phi(\Omega) \to L^\Phi(\Omega)$ is a composition operator, then

1. *$\|C_\tau\|_e = 0$ if and only if $\alpha = 0$.*

2. *$\|C_\tau\|_e \geq \alpha$ if $0 < \alpha \leq 1$ and $\Phi(x) \succ\succ x$.*

3. *$\|C_\tau\|_e \leq \alpha$ if $\alpha > 1$*

Proof.

1. From the above theorem we can conclude that C_τ is compact iff $\alpha = 0$.

2. Let $0 < \alpha \leq 1$ and $\Phi(x) \succ\succ x$.Let $0 < \epsilon < 2\alpha$ be arbitrary. Let $F = N(h, \alpha - \frac{\epsilon}{2})$, then by definition of α either F contains a non-atomic subset or has infinitely many atoms. If F contains a non-atomic subset then there are measurable subsets $E_n, n \in N$, such that $E_{n+1} \subseteq E_n \subseteq F, 0 < \mu(E_n) < \frac{1}{n}$. Let us define $f_n = \Phi^{-1}(1/\mu(E_n))\chi_{E_n}$. Then $\|f_n\|_\Phi = 1$ for all $n \in N$. We have to prove $f_n \to 0$ weakly.To prove this we have to show that $\int_\Omega f_n g \to 0$ for all $g \in L^\Psi(\Omega)$, where Ψ is the complementary function to Φ. Let $A \subseteq F$ with $0 < \mu(A) < \infty$ and $g = \chi_A$ since $\Phi(x) \succ\succ x$, then we have

$$|\int_\Omega f_n \chi_A d\mu| = \Phi^{-1}(\tfrac{1}{\mu(E_n)})\mu(A \cap E_n) \leq \Phi^{-1}(\tfrac{1}{\mu(E_n)})\mu(E_n) = \tfrac{\Phi^{-1}(1/\mu(E_n))}{1/\mu(E_n)} \to 0, n \to \infty$$

   Since simple functions are dense in $L^\Psi(\Omega)$, thus f_n is converge to 0 weakly. Now assume that F consists of infinitely many atoms. Let $(E_n)_{n=0}^\infty$ be disjoint atoms in F. Again on putting f_n as above. If $\mu(E_n) \to 0$, then by using the similar argument we had above, $\int_\Omega f_n \chi_A d\mu \to 0$. Now we have to prove that $\|C_\tau f_n\|_\Phi \geq \alpha - \frac{\epsilon}{2}$. Since $0 < \alpha - \frac{\epsilon}{2} < 1$ we have

$$\begin{aligned}
\|C_\tau f_n\|_\Phi &= inf\{\delta > 0 : \int_\Omega \Phi(\frac{|f_n \circ \tau|}{\delta}) d\mu \leq 1\} \\
&= inf\{\delta > 0 : \int_\Omega h\Phi(\frac{|f_n|}{\delta}) d\mu \leq 1\} \\
&\geq inf\{\delta > 0 : \int_\Omega (\alpha - \frac{\epsilon}{2})\Phi(\frac{|f_n|}{\delta}) d\mu \leq 1\} \\
&\geq inf\{\delta > 0 : \int_\Omega \Phi(\frac{(\alpha - \epsilon/2)|f_n|}{\delta}) d\mu \leq 1\} \\
&= (\alpha - \epsilon/2)\ inf\{\delta > 0 : \int_\Omega \Phi(\frac{|f_n|}{\delta}) d\mu \leq 1\} \\
&= \alpha - \frac{\epsilon}{2}
\end{aligned}$$

.

Finally let a compact Operator T on $L^\Phi(\Omega)$ such that $\|C_\tau - T\| < \|C_\tau\|_e + \frac{\epsilon}{2}$. Then we have

$$\begin{aligned}
\|C_\tau\|_e &> \|C_\tau - T\| - \frac{\epsilon}{2} \\
&\geq \|C_\tau f_n - T f_n\|_\Phi - \frac{\epsilon}{2} \\
&\geq \|C_\tau f_n\|_\Phi - \|T f_n\|_\Phi - \frac{\epsilon}{2} \\
&\geq (\alpha - \frac{\epsilon}{2}) - \|T f_n\|_\Phi - \frac{\epsilon}{2}.
\end{aligned}$$

for all $n \in N$. Since a compact operator maps weakly convergent sequences into norm convergent ones, it follows that $\|T f_n\|_\Phi \to 0$. Hence $\|C_\tau\|_e \geq \alpha - \epsilon$. Since ϵ is arbitrary, we obtain $\|C_\tau\|_e \geq \alpha$.

3. Let $\alpha > 1$ and take $\epsilon > 0$ be arbitrary and put $K = N(h, \alpha + \epsilon)$. The definition of α implies that K consist of finitely many atoms. Hence we can write $K = \{E_1, E_2, ..., E_m\}$

where $E_1, E_2, ..., E_m$ are distinct. Since $(M_{\chi_k} C_\tau f)(x) = \sum_{i=1}^{m} \chi_k(E_i) f(\tau(E_i))$, for all $f \in L^\Phi(\Omega)$, hence $M_{\chi_k} C_\tau$ has finite rank. Now, let $F \subseteq X\ K$ such that $0 < \mu(F) < \infty$, then we have

$$\mu \circ \tau^{-1}(F) = \int_F h d\mu \leq (\alpha + \epsilon)\mu(F).$$

Since $\alpha + \epsilon > 1$ and Φ^{-1} is a concave function, we obtain that

$$\Phi^{-1}(\tfrac{1}{\mu \circ \tau^{-1}(F)}) \geq \tfrac{1}{\alpha+\epsilon} \Phi^{-1}(\tfrac{1}{\mu(F)})$$

That is

$$\{\Phi^{-1}(\tfrac{1}{\mu \circ \tau^{-1}(F)})\}^{-1} \leq (\alpha + \epsilon)\{\Phi^{-1}(\tfrac{1}{\mu(F)})\}^{-1}.$$

It follows that $\|\chi_F \circ \tau\|_\Phi \leq (\alpha + \epsilon)\|\chi_F\|_\Phi$. Since simple functions are dense in $L^\Phi(\Omega)$, we obtain

$$sup_{\|f\|_\Phi \leq 1} \|\chi_{X/K} f \circ \tau\|_\Phi \leq \sup_{\|f\|_\Phi \leq 1} \|\chi_{x/k} f\|_\Phi \leq \alpha + \epsilon.$$

finally, since $M_{\chi_K} C_\tau$ is a compact operator, we get

$$\|C_\tau - M_{\chi_K} C_\tau = \sup_{\|f\|_\Phi \leq 1} \|(1 - \chi_k) C_\tau f\|_\Phi = \sup_{\|f\|_\Phi \leq 1} \|\chi_{x/k} C_\tau f\|_\Phi \leq \alpha + \epsilon.$$

□

Example 3.0.5. *Let Φ be an Orlicz function such that $\frac{\Phi^{-1}(2^n)}{2^n} \to 0$ as $n \to \infty$. Put Ω and μ as above. Define $\tau(1) = \tau(2) = \tau(3) = 1, \tau(4) = 2, \tau(5) = \tau(6) = 3, \tau(2n+1) = 5$, for $n \geq 3$, $\tau(2n) = 2n - 2$ for $n \geq 4$, and $\tau(x) = 5x$ for all $x \in (-\infty, 0]$. Then a simple function gives $h = 7/4_{\chi 1} + 1/4_{\chi 2} + 3/8_{\chi 3} + 1/3_{\chi 2n+1:n \geq 3} + 1/4_{\chi 2n:n \geq 4} + 1/5_{\chi(-\infty,0]}$, and $\alpha = \frac{1}{3}$. Thus $\|C_\tau\|_e \geq \frac{1}{3}$ on $L^\Phi(\Omega)$.*

3.1 Modular and norm continuity of composition operators

For the modular and norm continuity of composition operators C_τ in an Orlicz spaces $L^\Phi(\Omega)$, we represent necessary and sufficient conditions for any Orlicz function Φ and any $\sigma-$finite measure space (Ω, Σ, μ). For any Orlicz function Φ which satisfies Δ_2 condition for all u, the same is done for norm continuity of the composition operator C_τ in $L^\Phi(\Omega)$. If Φ satisfies Δ_2 condition for large u, then the problem of continuity of the composition operator C_τ in $L^\Phi(\Omega)$ is completely solved if the measure space is nonatomic of finite or infinite measure. Without any regularity condition on Φ, the conditions for continuity of C_τ from $L^\Phi(\Omega)$ into itself are explained in terms of the Radon-Nikodym derivative $\frac{d\mu\circ\tau^{-1}}{d\mu}$.

Theorem 3.1.1. *Assume that $\tau : \Omega \to \Omega$ is a measurable nonsingular transformation.*

1. *if $0 < a_\Phi = b_\Phi < \infty$ then $I_\Phi(C_\tau x) = I_\Phi(x)$ whenever $I_\Phi(x) < \infty$.*

2. *if $0 \le a_\Phi < b_\Phi \le \infty$ then the inequality*

$$I_\Phi(C_\tau x) \le K I_\Phi(x) \qquad (1)$$

holds for all x such that $I_\Phi(x) < \infty$ with some $K > 0$ independent of x if and only if

$$\mu(\tau^{-1}(A)) \le K\mu(A) \qquad (2)$$

for all $A \in \Sigma$ with $\mu(A) < \infty$.

Proof. 1. In this case the function Φ is 0 in the interval $[0,a_\Phi)$ and ∞ on (a_Φ, ∞). Therefore, $I_\Phi < \infty$ iff $\|x\|_\infty \le a_\Phi \Rightarrow \|C_\tau x\|_\infty \le a_\Phi \Rightarrow I_\Phi(C_\tau x) = 0 = I_\Phi$.

2. Let assume that $0 \le a_\Phi < b_\Phi \le \infty$.

   Necessary condition:

Let assume that the condition $I_\Phi(C_\tau x) \leq K I_\Phi(x)$ holds. If $A \in \Sigma$ and $\mu(A) = 0$,then non singularity of τ gives $\mu(\tau^{-1}(A)) = 0$ and we have $\mu(\tau^{-1}(A)) = K\mu(A)$. Thus suppose that $A \in \Sigma$ and $0 < \mu(A) < \infty$. Let $a \in (a_\Phi, b_\Phi)$ and taking $x = a\chi_A$. Then

$$I_\Phi = \int_A \Phi(a)d\mu(s) = \Phi(a)\mu(A) < \infty.$$

Since $C_\tau \chi_A = \chi_{\tau^{-1}(A)}$ then by (1) we have

$$\Phi(a)\mu(\tau^{-1}(A)) = I_\Phi(C_\tau x) \leq K I_\Phi(x) = K\Phi(a)\mu(A).$$

Since $0 < \Phi(a) < \infty$, then we have $\mu(\tau^{-1}(A)) \leq K\mu(A)$.

Sufficient condition:

Let assume that $0 \leq a_\Phi < b_\Phi \leq \infty$. and condition (2) satisfied. From this we have $\mu \circ \tau^{-1} \ll \mu$ by Radon-Nikodym theorem we have, $\mu \circ \tau^{-1}(A) = \int_A f_\tau(t)d\mu(t)$ for $A \in \Sigma$ and for some function f_τ locally integrable on Ω and $f_\tau \in L^\infty(\Omega)$ and $\|f_\tau\|_\infty \leq K$. Otherwise,there is $A \in \Sigma$ with $0 < \mu(A) < \infty$ such that $f_\tau(t) > K$ for any $t \in A$.This implies that $\mu \circ \tau^{-1}(A) = \int_A f_\tau(t)d\mu(t) > K\mu(A)$, which is a contradiction to (2).

Therefore,

$$\begin{aligned}
I_\Phi(C_\tau x) &= \int_\Omega \Phi(|C_\tau x(s)|)d\mu(s) \\
&= \int_\Omega \Phi(|x(\tau(s))|)d\mu(s) \\
&= \int_{\tau(\Omega)} \Phi(|x(t)|)d(\mu \circ \tau^{-1})(t) \\
&\leq \int_\Omega \Phi(|x(t)|)d(\mu \circ \tau^{-1})(t) \\
&= \int_\omega \Phi(|x(t)|)f_\tau(t)d\mu(t) \\
&\leq K\int_\Omega \Phi(|x(t)|)d\mu(t) \\
&= K I_\Phi(x).
\end{aligned}$$

□

Theorem 3.1.2. *Assume that $\tau : \Omega \to \Omega$ is a measurable nonsingular transformation. Then the composition operator C_τ is bounded from an Orlicz space $L^\Phi(\Omega)$ into itself, that is, there exists $M > 0$ such that*

$$\|C_\tau x\|_\Phi \leq M\|x\|_\Phi \textit{ for all } x \in L^\Phi(\Omega) \qquad (3)$$

If condition (2) *holds. If, in addition, Φ satisfies the condition Δ_2 for all $u > 0$,then* (3) *and* (2) *are equivalent.*

Proof. Necessary condition:

Let assume that condition (3) hold and by putting $x = \chi_A$ where $A \in \Sigma$ and $0 < \mu(A) < \infty$ we get

$$\frac{1}{\Phi^{-1}(1/\mu(\tau^{-1}(A)))} \leq \frac{M}{\Phi^{-1}(1/\mu(A))}$$
$$\Rightarrow \Phi^{-1}(\frac{1}{\mu(A)}) \leq M\Phi^{-1}(\frac{1}{\mu(\tau^{-1}(A))}) \qquad (4)$$

for all $A \in \Sigma$ with $0 < \mu(A) < \infty$

Since Φ satisfies Δ_2 condition for all $u > 0$,it follows that

$$L := sup_{u>0}\frac{\Phi(Mu)}{\Phi(u)} < \infty,$$

and $\Phi(Mu) \leq L\Phi(u)$ for all $u > 0$, which gives for $u = \Phi^{-1}(v)$ that

$$\Phi(M\Phi^{-1}(v)) \leq L\Phi(\Phi^{-1}(v)) \leq Lv$$

and so

$$M\Phi^{-1}(v) \leq \Phi^{-1}\{\Phi(M\Phi^{-1}(v))\} \leq \Phi^{-1}(Lv)$$

for all $v > 0$

From the condition (4) we get

$$\Phi^{-1}(\frac{1}{\mu(A)}) \leq M\Phi^{-1}(\frac{1}{\mu(\tau-1(A))}) \leq \Phi^{-1}(\frac{L}{\mu(\tau^{-1}(A))})$$

or equivalently $\mu(\tau^{-1}(A)) \leq L\mu(A)$ for all $A \in \Sigma$ with $0 < \mu(A) < \infty$, which finish the proof of the necessary condition with K=L.

Sufficient condition:

From Theorem 3.1.1 we know that if the condition (2) satisfied with $K \geq 1$ then condition (1) holds and

$$I_\Phi(\frac{C_\tau x}{K\|x\|_\Phi}) \leq \frac{1}{K} I_\Phi(\frac{C_\tau x}{\|x\|_\Phi}) \leq I_\Phi(\frac{x}{\|x\|_\Phi}) \leq 1.$$

hence $\|C_\tau x\|_\Phi \leq K\|x\|_\Phi$ for all $0 \neq x \in L^\Phi(\Omega)$ that is condition (3) hold with M=K. □

Remark 3.1.1. *The sufficient condition of theorem 3.1.2 can be proved in two different ways, namely by using simple functions and by the Orlicz interpolation theorem which is saying that any Orlicz space $L^\Phi(\Omega)$ is an exact interpolation space between $L^1(\Omega)$ and $L^\infty(\Omega)$.*

Remark 3.1.2. *Condition* (2) *is sufficient for the continuity of any composition operator from any symmetric space X into it self if X has either the Fatou property or an absolutely continuous norm, because X is then an interpolation space between L^1 and L^∞.*

Remark 3.1.3. *If $0 < a_\Phi \leq b_\Phi < \infty$, then the Orlicz spaces L^Φ is equal to L^∞ with an equivalent norm. Hence the composition operator C_τ is norm-continuous on $L^\Phi(\Omega)$ for every nonsingular transformation τ. However, in order to obtain modular continuity of C_τ which is stronger than norm continuity, we need the additional assumption* (2) *on τ as shown in the above theorem.*

3.2 Compact Composition Operators in Orlicz space

Theorem 3.2.1. *Let Φ be an Orlicz function and τ be a non singular measurable transformation from Ω to itself. Then the operator $C_\tau : L^\Phi(\Omega) \to L^\Phi(\Omega)$ is compact if and only if for any $\epsilon > 0$, the set $N(h, \epsilon)$=$\{x \in \Omega : h(x) > \epsilon\}$ consist of finitely many atoms.*

Proof. We shall prove it by method of contradiction. Let assume that $\epsilon > 0$ be given, the set N(h,ϵ) either contains a non-atomic subset or has infinitely many atoms. In both cases we can find a sequence of pairwise disjoint measurable subsets $\{A_n\}$ with $0 < A_n < \infty$ for every n. Let define f$_n = \Phi^{-1}(\frac{1}{\mu(A_n)})\chi_{A_n}$. Then I$_\Phi(f_n) = \int_\Omega \Phi(|f_n|)d\mu = \int_\Omega \Phi(\Phi^{-1}(\frac{1}{\mu(A_n)})\chi_{A_n})d\mu =$ 1,whence f$_n \in L^\Phi(\Omega)$ and $\|f_n\|_\Phi = 1$. So we have

$$\begin{aligned} I_\Phi(f_n \circ \tau) &= \int_\Omega \Phi(|f_n \circ \tau|)d\mu \\ &= \int_\Omega h\Phi(|f_n|)d\mu \\ &= \int_{A_n} \Phi(\Phi^{-1}(\frac{1}{\mu(A_n)})hd\mu \\ &\geq \epsilon \int_{A_n} \frac{1}{\mu(A_n)})d\mu \\ &= \epsilon \end{aligned}$$

whence $C_\tau \in L(L^\Phi(\Omega)$ and $\|C_\tau f_n\| \geq \epsilon$. Consiquently we have for m $\neq$ n:

$$\begin{aligned} I_\Phi(C_\tau f_m - C_\tau f_n) &= \int_\Omega \Phi(|f_m - f_n|)hd\mu \\ &= \int_{A_m} \Phi(|f_m|)hd\mu + \int_{A_n} \Phi(|f_n|)hd\mu \\ &= I_\Phi(f_m \circ \tau) + I_\Phi(f_n \circ \tau) \\ &\geq 2\epsilon \end{aligned}$$

Therefore, $\|C_\tau f_m - C_\tau f_n\|_\Phi \geq \epsilon$ for m,n $\in N$ with m $\neq$ n. This means that $\{C_\tau f_n\}$ contains no Cauchy subsequence, that is $C_\tau(U(L^\Phi(\Omega)))$ is not relatively compact, where $U(L^\Phi(\Omega))$ is the unit ball of $L^\Phi(\Omega)$. Consiquently, the operator C_τ is not compact, and so this is a contradiction.

Conversely, let $\epsilon > 0$ be given and the set N(h,ϵ) consists of finitely many atoms, $M_{\chi_{\tau^{-1}A}} C_\tau$

is a finite rank operator. Since h $< \epsilon$ on Ω A, for each f$\in L^{\Phi}(\Omega)$ with $\|f\|_{\Phi} \leq 1$ we have

$$\begin{aligned} I_{\Phi}(f \circ \tau - \chi_{\tau^{-1}A} f \circ \tau) &= I_{\Phi}((1 - \chi_{\tau^{-1}A}) f \circ \tau) \\ &= \int_{\Omega} |(\chi_{\Omega - A} \circ \tau) f \circ \tau| d\mu \\ &= \int_{\Omega - A} h\Phi(|f|) d\mu \\ &\leq \int_{\Omega - A} \epsilon\Phi(|f|) d\mu \\ &\leq \epsilon \|f\|_{\Phi} \leq \epsilon. \end{aligned}$$

it follows that

$$\|C_\tau - M_{\chi_{\tau^{-1}A}} C_\tau\| = sup\|C_\tau f - \chi_{\tau^{-1}A} C_\tau f\|_{\Phi} \leq \epsilon \text{ whenever } \|f\|_{\Phi} \leq 1.$$

Thus C_τ is the limit of some finite rank operators and is therefore compact. $\square$

Example 3.2.1. *Let Φ be an Orlicz function and let $\Omega = (-\infty, 0] \cup N$, where N is the set of natural numbers. Let μ be the Lebesque measure on $(-\infty, 0]$ and $\mu(n) = \frac{1}{2^n}$ if $n \in N$. Define $\tau : N \to N$ as $\tau(1) = 2$, $\tau(2) = \tau(3) = 3$,$\tau(4) = \tau(5) = \tau(6) = 4$ and $\tau(n) = n$ for $n \geq 7$, and $\tau(x) = \frac{2}{3x}$, for all $x \in (-\infty, 0]$. Direct computation shows that $h = 2\chi_{\{2\}} + 3\chi_{\{3\}} + \frac{7}{4\chi_{\{4\}}} + 1_{\chi\{n:n\geq 7\}} + \frac{3}{2_{\chi(-\infty,0]}}$,and $\alpha = \frac{3}{2}$.So $\|C_\tau\|_e \leq \frac{3}{2}$ on $L^{\Phi}(\Omega)$.*

Theorem 3.2.2. *Let $C_\tau \in \boldsymbol{B}(L^{\Phi}(\Omega))$. Then C_τ is compact if and only if $L^{\Phi}(\chi_\epsilon \mu\tau^{-1})$ is finite dimensional for each $\epsilon > 0$, where $\chi_\epsilon = \{x : \frac{d\mu\tau^{-1}}{d\mu}(x) \geq \epsilon\}$.*

Proof. Let $f \in L^{\Phi}(\Omega)$, then we have

$$\begin{aligned} \|C_\tau f\|_{\Phi} &= \inf\{\epsilon > 0 : \int \Phi(\frac{|f \circ \tau|}{\epsilon}) d\mu \leq 1\} \\ &= \inf\{\epsilon > 0 : \int \Phi(\frac{f}{\epsilon}) d\mu\tau^{-1}\} \\ &= \|f\|_{\Phi,\mu\tau^{-1}} \end{aligned}$$

Hence C_τ is a compact operator if and only if $I : L^\Phi(\chi_\epsilon \mu\tau^{-1}) \to L^\Phi(\chi_\epsilon \mu\tau^{-1})$ is compact operator if and only if $L^\Phi(\chi_\epsilon \mu\tau^{-1})$ is finite dimensional, where I is the identity operator. $\square$

Lemma 3.2.1. *If (Ω, Σ, μ) is a non-atomic measure space, then no non zero composition operator on $L^\Phi(\Omega)$ is compact.*

Lemma 3.2.2. *If (Ω, Σ, μ) is a $\sigma-$finite atomic measure space, then C_τ on $L^\Phi(\Omega)$ is compact if and only if the set $\{n : \sum_{m\in\tau^{-1}(a_n)} \mu(a_m) \geq \epsilon\mu(a_n)\}$ is a finite, where $a_1, a_2, ...$ are the atoms of the space.*

Theorem 3.2.3. *Let Φ be an Orlicz function vanishing only at zero with finite values, that is $a_\Phi = 0$ and $b_\Phi = \infty$. Let τ be a measurable nonsingular transformation from Ω into itself such that $\tau(\Omega) = \Omega$. If C_τ is a compact operator from L^Φ into itself, then the measure μ is purely atomic.*

Proof. Let $\Omega = \Omega_1 \cup \Omega_2$ where Ω_1 and Ω_2 are disjoint, let μ_{Ω_1} is non atomic and μ_{Ω_2} is purely atomic. Since $\mu\circ\tau^{-1} \ll \mu$, then by Radon-Nikodym theorem there exists a function h locally integrable on Ω_1 such that $\mu\circ\tau^{-1}(A) = \int_A h(t)d\mu$ for any $A \in \Sigma\cap\Omega_1$. Let define $A_0 = \{t \in \Omega_1 : h(t) > 0\}$. We have to prove that $\mu\circ\tau^{-1}(A_0) = 0$. We shall prove it by method of contradiction let assume that the statement is not true i.e. $\mu\circ\tau^{-1}(A_0) > 0$, then there exist a $\epsilon > 0$ such that the set $A_1 = \{t \in A_0 : h(t) \geq \epsilon\}$ has positive measure. Let us take a sequence B_n of pairwise disjoint subsets of $\Sigma\cap A_1$ where $0 < \mu(B_n) < 1/2^n$ for $n \in \mathbb{N}$. Let us define

$$x_n = \Phi^{-1}(1/\mu(B_n))\chi_{B_n} \text{ for } n > n_0$$

Thus $I_\Phi = 1$ when $x_n \in Ł^\Phi(\Omega)$ and $\|x_n\| = 1$ for $n > n_0$. For $m, n > n_0$ and $m \neq n$

$$\begin{aligned}
I_\Phi(C_\tau x_m - C_\tau x_n) &= \int_\Omega \Phi(|C_\tau x_m(s) - C_\tau x_n(s)|)d\mu(s) \\
&= \int_\Omega \Phi(|x_m(\tau(s)) - x_n(\tau(s))|)d\mu(s) \\
&= \int_{\tau(\Omega)} \Phi(|x_m(t) - x_n(t)|)d\mu \circ \tau^{-1}(t) \\
&= \int_\Omega \Phi(|x_m(t) - x_n(t)|)d\mu \circ \tau^{-1}(t) \\
&= \int_\Omega \Phi(|x_m(t) - x_n(t)|)h(t)d\mu(t) \\
&= \int_{B_m} \Phi(|x_m(t)|h(t)d\mu(t)) + \int_{B_n} \Phi(|x_n(t)|h(t)d\mu(t)) \\
&\geq \frac{1}{\mu(B_m)}\epsilon\mu(B_m) + \frac{1}{\mu(B_n)}\epsilon\mu(B_n) = 2\epsilon
\end{aligned}$$

Hence,$\|C_\tau x_m - C_\tau x_n\|_\Phi \geq 2\epsilon$ for $m, n > n_0$ and $m \neq n$. That means $\{C_\tau x_n\}$ contains no cauchy subsequence, that means $C_\tau(B(L^\Phi(\Omega)))$, where $B(L^\Phi(\Omega))$ is the unit ball of $L^\Phi(\Omega)$ is not compact.Hence C_τ is not compact which is a contradiction. Hence our assumption is wrong and $\mu \circ \tau^{-1}(A_0) = 0$. This complete the proof. $\square$

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